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NOTE

Numerical Evidence of Feigenbaum's Number δ in Non-linear Oscillations

1. INTRODUCTION

Chaotic behaviour and strange attractors in non-linear dynamics today are very intensively studied [1–3]. In practice, these new concepts arise especially in non-linear oscillations. Among many features, Feigenbaum's universal numbers [4] in relation to period-doubling bifurcations play a very important role in the study of a dynamical system on the route to chaos. Numerical methods for the computation of periodic solutions and period doubling bifurcations have been described in [5, 6].

Huberman and Crutchfield [7] and other authors [8–10] investigated the non-linear dynamics of particles in anharmonic potentials in the presence of an external periodic field. The equation of motion for the charge is given by

$$\frac{d^2x}{dt^2} + 0.4\frac{dx}{dt} + x - 4x^3 = 0.115 \cos \Omega t. \tag{1.1}$$

The non-linear characteristic is of the soft spring type and represented by a cubic term. A sequence of period doubling bifurcations has been noted when the forcing frequency Ω is close to 0.53. Räty *et al.* [8] found some estimates for the universal numbers to two decimal digits.

Recently, Thompson [11] studied the escape from a potential well. The differential equation which describes the motion of the particle is

$$\frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + x - x^2 = F \sin 0.85t, \tag{1.2}$$

the non-linearity being quadratic this time. A cascade of period doubling bifurcations has been found where the specific parameter now is the amplitude F of the periodic excitation. Transition to chaos occurs at a value of F close to 0.109.

The aim of this paper is to compute, to a very high precision, the transition values of the specific parameter at which period-doubling bifurcations take place. The suggested iterative technique uses the classical shooting method. The resulting non-linear equations are solved by the Newton method. The bifurcations are related to subharmonic solutions of the given system having the periods $2P$, $4P$, $8P$, ..., where P represents the period of the forcing term of the system. If p_i, where $i = 1, 2, 4, 8, \ldots$, denote the subsequent transition values, the Feigenbaum sequence deals with the sequence of the ratios $\Delta p_i/\Delta p_{2i}$, where $\Delta p_i = p_i - p_{2i}$. In the limit we obtain Feigenbaum's number

$$\delta = \lim_{i \to \infty} \frac{\Delta p_i}{\Delta p_{2i}} = 4.66920.... \tag{1.3}$$

2. THE ALGORITHM

We consider a periodic differential system written as

$$\frac{dx}{dt} = X(x, t), \tag{2.1}$$

where x and X are vectors of the same dimension. X has period P with respect to t and depends on some system parameter p. The period T of a subharmonic solution of (2.1) will be an integer multiple of P. Take some value of x, say x_0, as the initial point of the process related to time $t = 0$. This choice will be enlightened further in the sequel. We consider the image $f(x_0)$ of the point x_0, which is obtained by integrating numerically the system (2.1) with initial value x_0. Hereby the solution $x(t, x_0)$ is evaluated at the end of the period T. Thus, we have

$$f(x_0) = x(T, x_0). \tag{2.2}$$

We wish to find a periodic solution which is represented by a closed orbit. Therefore, we must determine a fixed point x^* of the map f

$$x^* = f(x^*). \tag{2.3}$$

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Let x_0 be an approximation to x^*. Define the correction vector as

$$\Delta x_0 = x^* - x_0, \tag{2.4}$$

and the error at the end of the integration as

$$r_0 = f(x_0) - x_0. \tag{2.5}$$

By linearizing (2.3) with respect to x_0 it is readily obtained that the correction vector satisfies

$$[I - A(T)]\, \Delta x_0 = r_0, \tag{2.6}$$

where I is the identity matrix. The matrix $A(t)$ is the fundamental matrix of the system of the first variational equations with respect to the reference solution $x(t, x_0)$ derived from (2.1), i.e.,

$$\frac{dy}{dt} = \frac{\partial X}{\partial x}\,[x(t, x_0), t]\, y. \tag{2.7}$$

Note from (2.6) that we must compute the fundamental matrix at the end of the period T. In practice, the numerical integration of (2.7) is performed simultaneously with that of the given system (2.1). The initial condition hereby is $A(0) = I$.

Now the integration rule must be chosen very carefully. From several numerical experiments it has been found that the use of the fourth-order Runge–Kutta method does not allow us to find the transition values of the specific parameter for the high periods in the period doubling cascade. Therefore, we suggest to make use of the Runge–Kutta–Hŭta method [12] which is a sixth-order eight-stage method showing very high accuracy. If h denotes the step length, the scheme is given by

$$x_{n+1} = x_n + \frac{h}{840}(41k_1 + 216k_3 + 27k_4 + 272k_5 + 27k_6 + 216k_7 + 41k_8), \tag{2.8}$$

where

$$\begin{aligned}
k_1 &= X[x_n, t_n],\\
k_2 &= X[x_n + \tfrac{1}{9}hk_1, t_n + \tfrac{1}{9}h],\\
k_3 &= X[x_n + \tfrac{1}{24}h(k_1 + 3k_2), t_n + \tfrac{1}{6}h],\\
k_4 &= X[x_n + \tfrac{1}{6}h(k_1 - 3k_2 + 4k_3), t_n + \tfrac{1}{3}h],\\
k_5 &= X[x_n + \tfrac{1}{8}h(-5k_1 + 27k_2 - 24k_3 + 6k_4), t_n + \tfrac{1}{2}h],\\
k_6 &= X[x_n + \tfrac{1}{9}h(221k_1 - 981k_2 + 867k_3 - 102k_4 + k_5), t_n + \tfrac{2}{3}h],\\
k_7 &= X[x_n + \tfrac{1}{48}h(-183k_1 + 678k_2 - 472k_3 - 66k_4 + 80k_5 + 3k_6), t_n + \tfrac{5}{6}h],\\
k_8 &= X[x_n + \tfrac{1}{82}h(716k_1 - 2079k_2 + 1002k_3 + 834k_4 - 454k_5 - 9k_6 + 72k_7), t_n + h].
\end{aligned} \tag{2.9}$$

Runge–Kutta methods of still higher order or even predictor–corrector methods may be chosen if necessary. This was not the case for the examples treated in the sequel.

We now return to the process defined by (2.6) which will be applied in an iterative manner. At each stage we must solve a linear system for the corrections Δx_0 using any classical method in numerical analysis. The iterative method is stopped if $|\Delta x_0|$ is sufficiently small. At each step in the iterative process the ameliorated value for x is now obtained from its previous value. In this way we obtain the closed orbit representing the periodic solution with the relevant period. The suggested iterative technique is based on the Newton method for solving non-linear equations. It is to be mentioned that the basic principles of the algorithm for the non-autonomous system (2.1) are the same as those described in Section 1 in [6]. Let us emphasize that P. Deuflhard [6] also develops a new Gauss–Newton algorithm for autonomous non-linear systems and a multiple shooting technique for dealing with special cases such as unstable periodic orbits or highly non-linear systems. These methods converge locally and quadratically. As is well known the subsequent transitions in the cascade of period doubling bifurcations occur as flip transitions. These are obtained as the transitions through the value -1 for one of the eigenvalues of $A(T)$. This is now a simple problem of polynomial interpolation in numerical analysis. Since the increments of the specific parameter are taken small in this continuation process, it is sufficient to choose a polynomial of low degree.

The whole procedure may now be described as follows. We take some value of the system parameter p for which there exists a $1P$ solution. Starting with an initial choice x_0 in the domain of attraction for the $1P$ solution we construct this periodic solution by solving (2.6) iteratively and we compute the eigenvalues of the matrix A. By taking values of the parameter p at regular small increments a continuation procedure is established for the $1P$ solution whereby each time (2.6) is solved and the eigenvalues of A are computed. As the starting value for x we take the solution obtained from solving (2.6) for the case with the last chosen value of the parameter p. The interpolation process related to the passage through the value -1 for one of the eigenvalues of A then determines the value of the parameter p at which the transition from the $1P$ solution to the $2P$ solution takes place.

Next we consider a value of the parameter p slightly beyond this transition value in the relevant domain of the $2P$ solution. In a similar way the continuation process for the $2P$ solution is performed and the value of p for the transition $2P \to 4P$ is computed from the interpolation process. The whole procedure now consists in repeating this process for all further desired transitions $4P \to 8P \to 16P \to 32P \to 64P \to \cdots$.

3. NUMERICAL RESULTS

Non-linear Oscillator with Cubic Term

With $x_1 = x$, $x_2 = \dot{x}$, (1.1) may be replaced by the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + 4x_1^3 - 0.4x_2 + 0.115 \cos \Omega t. \end{aligned} \tag{3.1}$$

The first variational equations for this system with respect to the reference solution $x_1(t)$, $x_2(t)$ are

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= (-1 + 12x_1^2)\, y_1 - 0.4y_2. \end{aligned} \tag{3.2}$$

We must calculate two solutions of (3.2) with initial conditions $y_1 = 1$, $y_2 = 0$, and $y_1 = 0$, $y_2 = 1$, respectively. Therefore, we perform simultaneously the numerical integration of the sixth-order system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + 4x_1^3 - 0.4x_2 + 0.115 \cos \Omega t, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= (-1 + 12x_1^2)\, x_3 - 0.4x_4, \\ \dot{x}_5 &= x_6, \\ \dot{x}_6 &= (-1 + 12x_1^2)\, x_5 - 0.4x_6, \end{aligned} \tag{3.3}$$

with the initial conditions at $t = 0$:

$$\begin{aligned} x_1 &= \bar{x}_{10}, \quad x_2 = \bar{x}_{20}, \quad x_3 = 1, \\ x_4 &= 0, \quad x_5 = 0, \quad x_6 = 1. \end{aligned} \tag{3.4}$$

Here the eigenvalues of $A(T)$ satisfy a quadratic equation.

Let us now study the occurrence of period doubling bifurcations and derive the corresponding Feigenbaum number. Let Ω_i, where $i = 1, 2, 4, 8, \ldots$, denote the value of the forcing frequency at which the transition takes place from a stable periodic solution with the period iP to an unstable periodic solution having the same period. At this Ω_i value one of the eigenvalues of the fundamental matrix $A(iP)$ passes through the value -1 and a new periodic solution is created with the period doubled $T = 2iP$. This new $2iP$ periodic solution

TABLE I

The continuation process of the $1P$ solution for the cubic term case

Ω	x_{10}	x_{20}	s_1	s_2	S = stable U = unstable
0.5372	0.006499784553	0.346556162311	0.950837 E-2	0.977372	S
0.5366	0.027738000952	0.342148345222	0.103635 E-1	0.892048	S
0.5360	0.039606299922	0.339097280182	0.125181 E-1	0.734649	S
0.5354	0.048162857953	0.336638752475	0.159799 E-1	0.572482	S
0.5348	0.055189321103	0.334457442699	0.225428 E-1	0.403683	S
0.5342	0.061281344923	0.332447534675	0.413219 E-1	0.219066	S
0.5336	0.066725980605	0.330557842780	0.045439 -0.083305 i	0.045439 +0.083305 i	S
0.5330	0.071687793585	0.328758908377	-0.041190 -0.085207 i	- 0.041190 +0.085207 i	S
0.5324	0.076271551668	0.327031901568	-0.407332 E-1	- 0.218725	S
0.5318	0.080548849547	0.325363901033	-0.211353 E-1	- 0.419302	S
0.5312	0.084571188412	0.323745571984	-0.144278 E-1	- 0.610966	S
0.5306	0.088377087499	0.322169901858	-0.109121 E-1	- 0.803496	S
0.5300	0.091996252845	0.320631458142	-0.873116 E-2	- 0.998831	S
0.5294	0.095452167632	0.319125926536	-0.724240 E-2	- 1.197698	U
0.5288	0.098763778494	0.317649809865	-0.616052 E-2	- 1.400469	U
0.5282	0.101946635922	0.316200224155	-0.533861 E-2	- 1.607377	U

TABLE II

The continuation process of the $4P$ solution for the cubic term case

Ω	x_{10}	x_{20}	s_1	s_2	S = stable U = unstable
0.528600	0.114615881255	0.309678295883	0.575892 E-8	0.954990	S
0.528575	0.115293259722	0.309318554720	0.690260 E-8	0.795970	S
0.528550	0.115749391666	0.309076597773	0.865041 E-8	0.634803	S
0.528525	0.116127663967	0.308876011602	0.116367 E-7	0.471457	S
0.528500	0.116461927077	0.308698769932	0.179182 E-7	0.305897	S
0.528475	0.116766789162	0.308537098536	0.396623 E-7	0.138090	S
0.528450	0.117050157405	0.308386791938	-0.171020 E-6	-0.031999	S
0.528425	0.117316891232	0.308245267213	-0.267457 E-7	-0.204403	S
0.528400	0.117570233980	0.308110801456	-0.144031 E-7	-0.379159	S
0.528375	0.117812480184	0.307982176494	-0.980979 E-8	-0.556301	S
0.528350	0.118045325332	0.307858492718	-0.740955 E-8	-0.735864	S
0.528325	0.118270065312	0.307739062942	-0.593476 E-8	-0.917884	S
0.528300	0.118487717488	0.307623347972	-0.493442 E-8	-1.102396	U
0.528275	0.118699098565	0.307510915210	-0.421795 E-8	-1.289437	U
0.528250	0.118904876210	0.307401411196	-0.367723 E-8	-1.479043	U

appears to be stable since both eigenvalues of $A(2iP)$ are in modulus smaller than 1.

We carried out the calculations on CDC CYBER 180/855 using single precision. The periods of interest are $T=P$, $2P$, $4P$, $8P$, $16P$, and $32P$, where P is the fundamental period of the excitation term, i.e., $P=2\pi/\Omega$. From numerical experiments the iterative process derived from (2.6) was stopped if $|\Delta x_0| \leqslant 10^{-12}$ and the step length h has been adapted accordingly.

In Tables I and II the continuation procedure is illustrated for, e.g., the $1P$ and the $4P$ solutions respectively when the forcing frequency Ω is slightly varied in the relevant domain of interest. The creation of such solutions occurs at $s_2=1$ and the transition to the next period-doubled solution takes place at $s_2=-1$.

In Table III we show the results for the transition values Ω_i of the forcing frequency and the initial conditions x_{10}, x_{20} for the relevant periodic solution in the phase plane. The passage through the value -1 for one of the eigenvalues of the fundamental matrix $A(iP)$ has been determined by the interpolation method mentioned at the end of Section 2 to six-digit accuracy. Note from Tables I and II that the $1P \to 2P$ and the $4P \to 8P$ transitions occur at $\Omega \approx 0.53$ and $\Omega \approx 0.52831$, respectively. Therefore, only a few additional smaller increments of the parameter Ω have been considered in the vicinity of these values. The final transition values for Ω are given to 15 digits to show the numerical convergence for Feigenbaum's number δ in the sequel.

We note from Table III that the transition values Ω_i tend to a limit value. It has been noted that at the last transition from $T=32P$ to $T=64P$ some precision is lost for the eigenvalue which passes through the value -1.

We now evaluate the ratios of the differences $\Delta\Omega_i/\Delta\Omega_{2i}$, where $\Delta\Omega_i = \Omega_i - \Omega_{2i}$. For the first four values of the Feigenbaum sequence we obtain

$$\begin{aligned} \delta_1 &= 4.73672, \\ \delta_2 &= 4.67939, \\ \delta_3 &= 4.67136, \\ \delta_4 &= 4.66902. \end{aligned} \tag{3.5}$$

This illustrates the numerical convergence to the Feigenbaum number $\delta = 4.66920$. We note that δ_4 shows an agreement of three decimal digits. For comparison, Räty *et al.* [8] mention the numerical value 4.8 as an estimate for δ_2. We should like to mention that the bifurcation tree in the amplitude versus forcing frequency representation has been given in [8] and therefore is not repeated here.

TABLE III

The transition values Ω_i and the initial conditions x_{10}, x_{20} for the cubic term case

T	Ω_i	x_{10}	x_{20}
P	0.529996440141057	0.092017217842	0.320622432432
2P	0.528607137070022	0.114062603600	0.309972874701
4P	0.528313831893278	0.118368119094	0.307686939068
8P	0.528251151712799	0.118194538377	0.307791604975
16P	0.528237733744171	0.113111963795	0.310534865395
32P	0.528234859914082	0.118034962355	0.307881539437

Non-linear Oscillator with Quadratic Term

For the problem of the escape from a potential well we must integrate the sixth-order system

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -x_1 + x_1^2 - 0.1x_2 + F\sin 0.85t,\\
\dot{x}_3 &= x_4,\\
\dot{x}_4 &= (2x_1 - 1)\,x_3 - 0.1x_4,\\
\dot{x}_5 &= x_6,\\
\dot{x}_6 &= (2x_1 - 1)\,x_5 - 0.1x_6.
\end{aligned} \tag{3.6}$$

The relevant parameter is now the amplitude F of the external periodic term. The fundamental period takes a fixed value $P = 2\pi/0.85$. The transitions $P \to 2P \to 4P \to 8P \to 16P \to 32P \to 64P$ have been studied.

The whole continuation process in the quadratic term case is similar to that for the cubic term case and therefore its complete tabular material is not reported for conciseness. Table IV gives the transition values F_i and the initial conditions x_{10} and x_{20}. The required precision was 12-digit accuracy for the iterative process from (2.6) to determine x_{10}, x_{20} and 9-digit accuracy for the interpolation method to derive the passage through the value -1 for one of the eigenvalues of $A(iP)$. For comparison, it is mentioned in [11] that the first transitions take place at $F_1 = 0.1005$ and $F_2 = 0.1073$. Thus the results are in excellent agreement. From Table IV the first four values in the Feigenbaum sequence are found to be

$$\begin{aligned}
\delta_1 &= 6.64186,\\
\delta_2 &= 5.06899,\\
\delta_3 &= 4.67802,\\
\delta_4 &= 4.66646.
\end{aligned} \tag{3.7}$$

Again the numerical convergence to Feigenbaum's number is obvious. However, compared to the previous example, it is slightly slower. We see that δ_4 is an approximation to Feigenbaum's universal number δ with an error of 2.8×10^{-3}.

In conclusion, these two examples on non-linear oscillations illustrate clearly that the suggested method based on the classical shooting method, in conjunction with the Newton method for solving the resulting non-linear equations, is very efficient to compute period-doubled solutions and its related Feigenbaum number.

TABLE IV

The transition values F_i and the initial conditions x_{10}, x_{20} for the quadratic term case

T	F_i	x_{10}	x_{20}
P	0.1005010010277272	-0.049299702056	-0.560867549711
2P	0.1073564130 74772	-0.217975510147	-0.475117758267
4P	0.1083885639 93247	-0.188776363276	-0.502018469114
8P	0.1085921846 19793	-0.201691489174	-0.491789268742
16P	0.1086357111748362	-0.197082998579	-0.495716936092
32P	0.108645039409791	-0.199020470917	-0.494119373771

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